International Institute of Information Technology, Hyderabad

(Deemed to be University)

MA4.101-Real Analysis (Monsoon-2025)

Assignment 1

Due date: **October 10, 2025** Total Marks: 80

Disclaimer
The level of depth and difficulty in this assignment is **not indicative of the upcoming mid-semester exam**. Its purpose is to provide practice with key concepts covered in all three sections.

Question (1) [10 Marks] Let $(x_n)_{n\geq 1}$ be a bounded sequence of real numbers and let

$$y_n := \frac{x_{n+1}}{n}, \qquad n \geq 1.$$

(a) [**5 Marks**] Does $(y_n)_{n\geq 1}$ necessarily converge? If yes, find its limit and prove it.

(b) [**5 Marks**] If not (in a more general setting where (x_n) might be unbounded), give a natural additional condition on (x_n) which guarantees $(y_n) \to 0$.

Question (2) [10 Marks] Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be real sequences. Define the interleaved sequence $(c_n)_{n\geq 1}$ by

$$c_{2n} = a_n, \qquad c_{2n+1} = b_n \qquad (n \geq 1).$$

(a) [**5 Marks**] If $a_n \to L$ and $b_n \to L$, prove that $c_n \to L$.

(b) [**5 Marks**] If instead $a_n \to L_1$ and $b_n \to L_2$ with $L_1 \neq L_2$, determine $\limsup c_n$ and $\liminf c_n$ (and justify your answer).

Question (3) [10 Marks] Let $(x_n)_{n\geq 1}$ be a bounded sequence of real numbers and define its *cluster-radius* by

$$R(x_n) \;:=\; \limsup_{n\to\infty} x_n \;-\; \liminf_{n\to\infty} x_n.$$

Answer the following.

(a) [**2 Marks**] Prove that $R(x_n) = 0$ if and only if the sequence (x_n) converges.

(b) [**3 Marks**] Let $(y_n)_{n\geq 1}$ be another bounded real sequence. Prove the inequality

$$R(x_n + y_n) \;\leq\; R(x_n) \;+\; R(y_n).$$

(c) [**5 Marks**] Give one example where equality holds in part (b), and one example where the inequality is strict. For each example justify your claim.

Question (4) [10 Marks] Let X and Y be nonempty sets and let

$$f : X \times Y \longrightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}.$$

(a) [**4 Marks**] Prove that

$$\sup_{y\in Y}\, \inf_{x\in X} f(x,y) \;\leq\; \inf_{x\in X}\, \sup_{y\in Y} f(x,y).$$

(b) [**6 Marks**] Prove the *principle of iterated suprema/infima*:

$$\sup_{(x,y)\in X\times Y} f(x,y) \;=\; \sup_{x\in X}\sup_{y\in Y} f(x,y) \;=\; \sup_{y\in Y}\sup_{x\in X} f(x,y),$$

and

$$\inf_{(x,y)\in X\times Y} f(x,y) \;=\; \inf_{x\in X}\inf_{y\in Y} f(x,y) \;=\; \inf_{y\in Y}\inf_{x\in X} f(x,y).$$

Question (5) [10 Marks] Let

$$U = \Big\{ m + \frac{1}{n} : m, n \in \mathbb{Z},\; n \neq 0 \Big\}.$$

(a) **[3 Marks]** Show that U is countable.

(b) **[4 Marks]** Prove that U is dense in $\mathbb{R}$.

(c) **[3 Marks]** Despite being dense, explain why U has no limit point outside $\mathbb{Q}$.

Question (6) [10 Marks] Let us introduce a new kind of number system $\mathbb{D}$. Every number in this system looks like $a + b\varepsilon$, where $a, b \in \mathbb{Q}$ and ε is a special new symbol with the rule $\varepsilon^2 = 0$ but $\varepsilon \neq 0$. Here $b\varepsilon$ means "b multiplied by ε". In other words

$$\mathbb{D} = \left\{a + b\varepsilon \ : \ a, b \in \mathbb{Q}, \ \varepsilon \neq 0 \text{ and } \varepsilon^2 = 0\right\}.$$

Addition and multiplication are defined by

$$(a + b\varepsilon) + (c + d\varepsilon) = (a + c) + (b + d)\varepsilon,$$
$$(a + b\varepsilon)(c + d\varepsilon) = ac + (ad + bc)\varepsilon.$$

Answer the following.

(a) **[2 Marks]** Find a special number $e_+ \in \mathbb{D}$ such that $e_+ + y = y$ for all $y \in \mathbb{D}$. Find another special number $e_\times \in \mathbb{D}$ such that $ye_\times = y$ for all $y \in \mathbb{D}$.

(b) **[2 Marks]** For a given number $x = a + b\varepsilon \in \mathbb{D}$ with $a \neq 0$ find another number $z \in \mathbb{D}$ such that $xz = e_\times$.

(c) **[2 Marks]** Show that there exist nonzero numbers in $\mathbb{D}$ such that their square is 0.

(d) **[4 Marks]** Let $a + b\varepsilon, c + (-d\varepsilon) \in \mathbb{D}$, where $c \neq 0$. Compute $\frac{(a+b\varepsilon)^n}{(c-d\varepsilon)^m}$.

Question (7) [10 Marks] Consider the following weaker notion of convergence: we say that a sequence $(x_n)_{n\geq 1}$ of real numbers *weakly converges* to L if for every $\varepsilon > 0$, there exists $N \geq 1$ such that

$$|x_n - L| < \varepsilon \text{ holds for infinitely many } n \geq N.$$

In words, beyond some point, infinitely many terms come arbitrarily close to L. Answer the following.

(a) **[2 Marks]** Show that every (usual) convergent sequence also weakly converges to the same limit.

(b) **[3 Marks]** Give an explicit example of a bounded sequence that weakly converges but does not converge in the usual sense.

(c) **[3 Marks]** Does the weak limit, if it exists, have to be unique? Either prove uniqueness or exhibit a counterexample.

(d) **[2 Marks]** Briefly explain why the standard definition of convergence requires *all sufficiently large terms* to stay close to the limit, rather than only infinitely many. What property of limits would fail otherwise?

Question (8) [10 Marks] Let (a_n) be defined recursively by

$$a_1 = 2, \qquad a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right), \quad n \geq 1.$$

(a) **[3 Marks]** Show that $(a_n)_{n\geq 1}$ is bounded below by $\sqrt{3}$.

(b) **[3 Marks]** Prove that $(a_n)_{n\geq 1}$ is decreasing.

(c) **[4 Marks]** Compute $\lim\limits_{n\to\infty} a_n$.